

# **Monotone Preferences over Information**

**By**

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# Monotone Preferences over Information\*

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## Abstract

We consider preference relations over information that are monotone: more information is preferred to less. We prove that, if a preference relation on information about an uncountable set of states of nature is monotone, then it is not representable by a utility function.

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# 1 Introduction

Understanding the value of information has been in the minds of economists and statisticians for a long time. There is an older literature (e.g. Blackwell (1951), Marschak (1974), Gould (1974), and Allen (1983)), and a renewed interest in the value of information in recent years (e.g. Athey and Levin (1998), Persico (1996), and Persico (1999)).

In this paper we make three contributions to this literature. First we prove two impossibility theorems. We consider preference relations over information that are monotone, in the sense that more information is strictly preferred to less; we show that, if the state space is uncountable, no monotone preference relation over information can be represented by a utility function. The two theorems account for the two usual ways of modeling information: through partitions of the state space, and through sigma algebras.

Our result is important because it shows that utility theory is not likely to be a useful tool in the analysis of the value of information. This finding should be contrasted with the existing literature on the value of information, where utility representations are used. The use of a utility implies that preferences are not monotone.

Our second contribution is didactic. We give a simple proof of one of our impossibility theorems when the state space is  $[0,1]$ . We believe that this is a better example of non-representability than the usual textbook example, lexicographic preferences. Lexicographic preferences are not present in many economic applications, while problems involving the value of information are common. Our method of proof is essentially the same as that of the standard textbook proof of non-representability of lexicographic preferences.

Our final contribution is to show that monotone preferences over information are the first economic example of non-representability that is essentially different from lexicographic preferences. Recently, Beardon, Candeal, Herden, Induráin and Mehta (2000) have shown that there are exactly four classes of non-representable preferences, one of which is the set of preferences that are isomorphic to lexicographic preferences. Beardon et al. (2000) argue that all economic examples of non-representability belong to the lexicographic class; we show that monotone preferences over information belong to one of the other three classes (concretely, it is a long line, see below for a definition).

The maintained assumption in the paper is that preferences are monotone (as in Blackwell's (1951) order). A decision maker (DM) that conforms to Savage's axioms, and thus has priors over the state space, will typically not have monotone preferences over information. To see this, suppose that the state space is the interval  $[0,1]$ , and that DM's priors are represented by the uniform distribution. Then, DM is indifferent between total ignorance and receiving a signal that tells her if the state  $1/2$  has occurred or not. Ex-post knowledge of the state  $1/2$  may be valuable, but since it is a probability zero event the signal is worthless to DM.

Still, we believe that monotonicity is a natural assumption for at least two reasons. First, it is not clear that most people have an understanding of probability zero events; it is dubious that, if asked, many people would be *exactly* indifferent between ignorance and the  $1/2$ -signal above. It is, after all, an empirical question: what is the best behavioral

assumption for the analysis of information, Savage’s axioms or monotonicity? The stage is indeed set for a “paradox”, if people make monotone choices over information they cannot have priors.

Second, the problem of whether an individual likes finer partitions is independent of, and maybe more basic than, whether DM’s preferences accord with Savage’s theory. We may wish to analyze the robustness of a utility representation, in which case we need to analyze arbitrary preferences over information, and representation breaks down. In fact, representation rests on a large number of indifferences; any psychological wrinkle that could tilt this indifferences towards monotonicity makes any utility representation break down. In a vague sense, then, representable preferences over information are non-generic.

To illustrate this point, we show how monotone preferences arise naturally if the individual is a maxminimizer. Since it has been argued that this may happen if DM is uncertainty averse, the experimental evidence that individuals dislike uncertainty suggests that monotone preferences may be empirically important.<sup>1</sup>

## 2 The Non-representation Theorems

One strategy for modeling information is to identify information with partitions of the state space; in this model, more information is a finer partition. In mathematics, this approach was initiated by Hintikka (1962), and introduced to economics by Aumann (1974). A preference relation on the set of all partitions is monotone if finer partitions are preferred to coarser partitions. The second approach is to model an agent’s information by a  $\sigma$ -algebra over the state space—this approach is common in statistics, but also in economics and finance. A  $\sigma$ -algebra represents more information than another  $\sigma$ -algebra if it is finer. Monotone preferences on the set of  $\sigma$ -algebras is monotone if, whenever one  $\sigma$ -algebra is contained in another, the larger one is preferred.

In this section we prove the main results of this paper: that monotone preferences over information can not be represented by a utility function if the state space is uncountable. In the next subsection we prove the result for the partitions approach, in Theorem 1. In the following subsection we prove it for the  $\sigma$ -algebra approach, in Theorem 2.

Theorems 1 and 2 are independent results, as the two approaches to modeling information are not equivalent, and neither model is more general than the other (see Dubra and Echenique (2000)).

### 2.1 Partitions

In this section we model information by partitions of a set of possible states of nature,  $\Omega$ . A **partition**  $\tau$  of  $\Omega$  is a collection of pairwise disjoint subsets whose union is  $\Omega$ ; note that for each state of nature  $\omega$  there is a unique element of  $\tau$  that contains  $\omega$ . A decision maker whose information is represented by  $\tau$  is informed only that the element of  $\tau$  that

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<sup>1</sup>See Gilboa and Schmeidler (1989) where the relation between uncertainty aversion and maxmin preferences is discussed.

contains the true state of nature has occurred. In other words, the decision maker cannot distinguish between states that belong to the same element of  $\tau$ .

A **preference relation** on a set  $X$  is a complete (total), transitive binary relation on  $X$ . A preference relation  $\preceq$  is **representable** if there is a function  $u : X \rightarrow \mathbf{R}$  such that  $x \preceq y$  if and only if  $u(x) \leq u(y)$ .

Let  $\mathcal{P}(\Omega)$  be the set of all partitions of  $\Omega$ . If  $\tau, \tau' \in \mathcal{P}(\Omega)$ , say that  $\tau'$  is **finer** than  $\tau \neq \tau'$  if, for every  $A \in \tau'$ , there is  $B$  in  $\tau$  such that  $A \subseteq B$ . A preference relation  $\preceq$  on  $\mathcal{P}(\Omega)$  is **monotone** if  $\tau \prec \tau'$  whenever  $\tau'$  is finer than  $\tau$ .

Monotonicity is a natural assumption on preferences: if  $\tau'$  is a finer partition than  $\tau$ , then  $\tau'$  contains more information.<sup>2</sup> The intuition is the following. Suppose a decision maker has information represented by  $\tau'$ . When state  $\omega$  occurs, she is informed of the event  $B \in \tau'$ . That is, she knows that some state in  $B$  has happened, but does not know which one exactly. If her information had been represented by  $\tau$ , she would have known that a certain event  $A \supseteq B$  occurred. In this case, she could not rule out states in  $A$  but not in  $B$ , whereas, if her partition is  $\tau'$ , she would know that states in  $A \setminus B$  did not occur.

We now state our main theorem. It establishes that when the state space is uncountable, preferences that prefer more information to less cannot be represented by a utility function.

**Theorem 1** *Let  $\Omega$  be uncountable. If  $\preceq$  on  $\mathcal{P}(\Omega)$  is monotone then it is not representable.*

**Remark.** Although all the theorems in this note are stated for complete preorders, the proofs show that the theorems hold for possibly incomplete preorders. For example, Theorem 1 would say that if an incomplete preference relation is monotone, there does not exist a representable proper extension.<sup>3</sup>

All proofs, except that of Proposition 3, are presented in the appendix. It suffices here to say that the reason why the theorem is true, is that there are more “information structures” (partitions) than real numbers and monotonicity limits the “amount” of indifference the individual can have.

## 2.2 $\sigma$ algebras

In statistics and finance, but also in economics (see for example (Allen, 1983)), information is often modeled as  $\sigma$ -algebras on the space of states of nature, instead of using partitions. In this model, there is a primitive measurable space  $(\Omega, F)$ , and information is identified with sub- $\sigma$ -algebras of  $F$ .

Let  $(\Omega, 2^\Omega)$  be the primitive measurable space. Let  $\mathcal{F}(\Omega)$  be the set of all  $\sigma$ -algebras on  $\Omega$ . If  $F, G \in \mathcal{F}(\Omega)$ , say that  $F$  is **finer** than  $G$  if  $G$  is a proper subset of  $F$ , noted

<sup>2</sup>Which does not contradict that  $\tau'$  could have more information than  $\tau$  and not be finer, only that refinement is sufficient for more information. So, our definition does not contradict the analysis in Athey and Levin (1998)

<sup>3</sup>A preorder  $\leq$  is a proper extension of  $\preceq$  if  $p \prec q$  implies  $p < q$ . See Dubra and Ok (2000).

$G \subsetneq F$ . The intuition behind the use of  $\sigma$ -algebras is that if  $A, B \subseteq \Omega$  are not measurable but  $A \cup B$  is, then the decision maker cannot distinguish between states in  $A$  and states in  $B$ ; she can distinguish between states in  $A \cup B$  and in  $(A \cup B)^c$ . Thus if  $F$  is finer than  $G$ , then  $F$  represents more information than  $G$ .

A preference relation  $\preceq$  on  $\mathcal{F}(\Omega)$  is **monotone** if  $G \prec F$  whenever  $F$  is finer than  $G$ .

**Theorem 2** *Let  $\Omega$  be uncountable. If  $\preceq$  on  $\mathcal{F}(\Omega)$  is monotone then it is not representable.*

## 2.3 Theorem 1 junior grade

Theorems 1 and 2 show that utility theory is not a useful tool in the analysis of the value of information. Besides this substantive contribution, we can also make a didactic contribution by providing a simple example of non-representability. The canonical example of non-representability is lexicographic preferences, but the only place students of economics find lexicographic preferences is in discussions of representability. We believe that preferences over information is a more relevant example of non-representability. Proposition 3 shows that no monotone preference over information on partitions of  $[0, 1]$  is representable. The method of proof is basically the same as for lexicographic preferences.

**Proposition 3** *Let  $\Omega = [0, 1]$ . If  $\preceq$  on  $\mathcal{P}(\Omega)$  is monotone then it is not representable.*

**Proof.** Suppose, by way of contradiction, that there is a function  $u : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$  that represents  $\preceq$ . For each  $x \in (0, 1)$  let

$$\begin{aligned}\tau_x &= \{\{y\} : 0 \leq y < x\} \cup [x, 1], \\ \tau'_x &= \{\{y\} : 0 \leq y \leq x\} \cup (x, 1].\end{aligned}$$

Note that  $\tau_x, \tau'_x \in \mathcal{P}(\Omega)$ , and that  $\tau_x \prec \tau'_x$ , as  $\tau'_x$  is finer than  $\tau_x$ . But then there is a rational number  $r(x)$  such that  $u(\tau_x) < r(x) < u(\tau'_x)$ . Let  $x \neq \tilde{x}$ , say  $x < \tilde{x}$ , then  $\tau_{\tilde{x}}$  is finer than  $\tau'_x$ . Thus

$$u(\tau_x) < r(x) < u(\tau'_x) < u(\tau_{\tilde{x}}) < r(\tilde{x}) < u(\tau'_{\tilde{x}}).$$

But then  $r$  is injective, a contradiction. ■

**Remark.** Non-representability in general uncountable subsets of  $\mathbf{R}$  can be proven by a slight modification of the proof of proposition 3.

## 3 Recovering the representation.

Given the negative result in Theorem 1, one may wonder under what conditions one can recover a utility representation for preferences for information. In this section we discuss the existence of a representation when  $\Omega$  is countable, and then present two alternative models that yield a representation, and comment on their relative merits.

In what follows we will only deal with the partitions model because we believe that this is the more natural way to model information.

### 3.1 Countable $\Omega$

It is natural to ask if Theorem 1 can be strengthened to countable  $\Omega$ . Example 4 shows that it can not. When  $\Omega$  is countable, there are monotone preferences over information that are representable. Example 4 may be somewhat misleading, though. We show (Theorem 5) that, if preferences are monotone, but individual states are still relatively unimportant, then there is a utility if and only if  $\Omega$  is finite.

**Example 4** Consider  $\Omega = \{(1/2)^i : i \in \mathbf{N}\}$ . Any  $\tau \in \mathcal{P}(\Omega)$  has at most a countable number of elements, say  $\tau = \{A_k : k \in \mathbf{N}\}$  (if  $\tau$  has a finite number of elements, put  $A_k = \emptyset$  as often as necessary). Let  $u(\tau) = \sum_{k=1}^{\infty} \inf A_k$ . Then  $u$  represents a monotone preference relation over information on  $\Omega$  (namely the preference relation induced by  $u$ ).

Let  $\Omega$  be a set and  $\preceq$  a preference relation on  $\mathcal{P}(\Omega)$ . An element  $\omega \in \Omega$  is an **atom** for  $\preceq$  if, for any  $A \subseteq \Omega$  with  $\omega \in A$  and at least two elements,

$$\{A, A^c\} \preceq \tau \preceq \{\{\omega\}, A \setminus \{\omega\}, A^c\}$$

is satisfied only for  $\tau = \{A, A^c\}$  or  $\tau = \{\{\omega\}, A \setminus \{\omega\}, A^c\}$ . A state of nature is an atom if the decision maker gains relatively little from being perfectly informed about this state—in the sense that any partition that is preferred over  $\{A, A^c\}$  is also preferred over  $\{\{\omega\}, A \setminus \{\omega\}, A^c\}$ .

**Theorem 5** Let  $\Omega$  be a set. A monotone preference relation on  $\mathcal{P}(\Omega)$  that has an atom is representable if and only if  $\Omega$  is finite.

### 3.2 Priors on $\Omega$ and worthless states.

We argued in the Introduction that the existence of priors on the set of states of nature could imply that preference are not monotone. We present a simple model where a utility representation for partitions arises. Versions of this model are used in many papers on the value of information (e.g. Blackwell (1951) and Athey and Levin (1998)).

There is a set  $\Omega$  of states of nature. DM must choose an action, an element in a set  $A$ , after observing a signal about the state of nature. DM's prior knowledge is represented by the probability measure  $\mu$  over  $\Omega$ , given a probability space  $(\Omega, F, \mu)$ . Let  $u : \Omega \times A \rightarrow \mathbf{R}$  be DM's (measurable) state-contingent utility function.

Given any partition  $\tau \in \mathcal{P}(\Omega)$  and  $\omega \in \Omega$ , let  $k_\tau(\omega) \in F$  be the element of  $\tau$  that contains  $\omega$ . When  $\omega$  is realized, the decision maker is informed that an element in  $k_\tau(\omega)$  has occurred. Let

$$a^*(\omega) \in \arg \max_{a \in A} \int_{k_\tau(\omega)} u(\tilde{\omega}, a) d\mu(\tilde{\omega}),$$

so that for each  $\omega$ ,  $a^*(\omega)$  is DM's optimal choice, given her signal  $k_\tau(\omega)$  (in fact the selection  $a(\cdot)$  can be taken to be measurable). Then a utility function  $U$  on  $\mathcal{P}(\Omega)$  is defined by

$$U(\tau) = \int_{\Omega} \left\{ \int_{k_\tau(\omega)} u(\tilde{\omega}, a^*(\omega)) d\mu(\tilde{\omega}) \right\} d\mu(\omega).$$

To see why the resulting preference relation is not monotone, let all singleton sets be measurable (i.e.  $\{\omega\} \in F$  for all  $\omega \in \Omega$ ) and note that all but a countable number of  $\omega$  have zero probability. Then, since it is worthless to be perfectly informed in a zero probability event, DM's utility is not higher after a refinement of a zero probability  $\omega$ . Thus, requiring that DM has priors is like reducing the size of  $\Omega$ .

Note that the construction of  $U$  requires a good deal of faith in the setup. If we wish to analyze the robustness of the  $U$  construction we would need to consider preferences over  $\mathcal{P}(\Omega)$ , and representation is no longer guaranteed.

### 3.3 Finite Action Space.

The value of information, of a finer partition, is that DM has less restrictions on her choice of action. DM must choose the same action at states that she cannot distinguish, so a finer partition eases some restrictions and thus must make DM (weakly) better off. If DM faces a limited number of alternative actions, more information may not always make a difference—DM will not strictly gain from more information. Thus, a limit on the number of possible choices has much the same effect as the existence of priors, it limits the value of being informed in particular states.

Again,  $\Omega$  is the set of states of nature and DM must choose an action in  $A$  after observing a signal about the state of nature.

The primitives of the model are a collection  $\{\preceq_B\}_{B \in 2^\Omega}$  of preference relations over  $A$ , and a preference relation  $\preceq$  over  $\mathcal{P}(\Omega)$ . The interpretation of  $\preceq_B$  for a fixed subset  $B$  of  $\Omega$  is the following. Suppose state  $\omega$  occurs and DM is informed of the element of the partition that has occurred, say  $B = k_\tau(\omega)$ . Given this, she chooses an action that is maximal according to  $\preceq_B$ . Thus, each partition  $\tau$  generates a function  $a_\tau : \Omega \rightarrow A$ . Let  $f : \mathcal{P}(\Omega) \rightarrow A^\Omega$  be the map that takes partitions into functions from  $\Omega$  to  $A$ :  $f(\tau) = a_\tau$ . In addition, DM is endowed with the preference relation  $\preceq$  which is assumed to be consistent with the collection  $\{\preceq_B\}$  in the sense that, if two partitions  $\tau$  and  $\tau'$  are such that  $a_\tau = a_{\tau'}$ ,  $\tau \sim \tau'$ .

The next proposition shows, as was argued in the beginning of this section, that reducing the number of actions DM can adopt enables representation of her preferences.

**Proposition 6** *If  $\Omega$  is a compact metric space,  $A$  is finite, and  $f(\omega)$  is continuous for all  $\omega \in \Omega$ , then  $\preceq$  is representable.*

## 4 Preferences over information are not lexicographic

“So the answer to the crucial question in utility theory about whether or not the only non-representable preference relation is essentially the Debreu (lexicographic) chain is, somewhat informally, yes provided that we do not want examples based on ordinal numbers with large cardinality.” Beardon et al. (2000)



Theorem 7 below shows that a monotone preference over an uncountable state space is essentially different from lexicographic preferences. As was shown in Proposition 3 one can build an example of a non representable preference relation that makes no explicit use of ordinal numbers. Still, of course, the reason why representability fails is the large cardinality of the set of all partitions on  $\Omega$  : non-representability in Theorem 1 comes from the existence of too many partitions to be ranked strictly. The existence of a utility would imply that there are “only” a continuum many partitions that can be strictly ranked.

The **dual** order of a given ordered set  $\langle X, \preceq \rangle$  is the order  $\preceq_d$  on  $X$  defined by  $x \prec_d y$  if and only if  $y \prec x$ . Let  $\gamma$  be the first uncountable ordinal. An ordered set  $\langle X, \preceq \rangle$  is **long** if  $\langle X, \preceq \rangle$ , or  $\langle X, \preceq_d \rangle$ , contain a sub-chain which is order-isomorphic to  $[0, \gamma)$ .<sup>4</sup>

**Theorem 7** *Let  $\Omega$  be uncountable. If  $\preceq$  on  $\mathcal{P}(\Omega)$  is monotone then  $\langle \mathcal{P}(\Omega), \preceq \rangle$  is long.*

## 5 An example: Maxmin Preferences

We present an example of a decision problem with maxmin preferences. Under our assumptions, the derived value of information is such that being informed in a particular state makes DM always strictly better off. Because of this monotonicity, her preferences are not representable by a utility.

Let  $\Omega = [0, 1]$  and  $P$  a set of probability measures on  $\Omega$ . DM must choose an element (action) in  $A = [0, 1]$  after observing a signal about the state of nature. Her state-contingent utility is given by  $u(\omega, a)$ , with  $u(\omega, a) < u(\omega, \omega)$  for all  $a \neq \omega$ . We will assume that DM is a maxminimizer, so the utility in event  $B$  when action  $a$  is chosen is

$$U(B, a) = \inf_{p(B) > 0} \int_B \frac{u(\tilde{\omega}, a)}{p(B)} dp(\tilde{\omega}).$$

We need  $\max U(B, a)$  to be well defined, so that  $a[B]$ , the optimal action in event  $B$  exists. For example, if  $u(\omega, a) = -(\omega - a)^2$  (DM is a statistician seeking to minimize the mean squared error), and  $P$  contains all degenerate priors on  $\Omega$ , then  $\max U(B, a)$  is well defined. To see this, let  $\overline{B}$  stand for the closure of  $B$ , and  $a_{B,a} \in \arg \max_{\omega \in \overline{B}} (\omega - a)^2$ , we have that  $U(B, a) = u(a_{B,a}, a)$ . Therefore,  $U(B, a)$  is a continuous function of  $a$ , and  $a[B]$ , the optimal action in event  $B$  is well defined.

A set  $P$  of probability measures over  $\Omega$  is **broad** if the set of  $\omega \in \Omega$  such that  $p(\omega) > 0$  for some  $p \in P$  is uncountable. This is the case, for example, if  $P$  contains all degenerate probability measures.

We will assume that DM’s preferences over partitions satisfy the following axiom.

**Dominance.** If for all  $\omega \in \Omega$ ,

$$U(k_\tau(\omega) \cap k_{\tau'}(\omega), a[k_\tau(\omega)]) \geq U(k_\tau(\omega) \cap k_{\tau'}(\omega), a[k_{\tau'}(\omega)])$$

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<sup>4</sup>See Beardon et al. (2000) for details.

and there exists  $\tilde{\omega}$  and  $p \in P$  with  $p(\tilde{\omega}) > 0$  such that the above inequality is strict, then  $\tau \succ \tau'$ .

DM is comparing two partitions  $\tau$  and  $\tau'$ . In doing so, he imagines himself in a fixed event  $k_\tau(\omega) \cap k_{\tau'}(\omega)$ . Suppose he realizes that the utility he would obtain by choosing the optimal action in any of the states in that event is weakly larger under  $\tau$  than under  $\tau'$ . Suppose in addition that DM believes that, with positive probability, a state will occur in which, choosing the optimal action under  $\tau$  will make him strictly better off than choosing the optimal action under  $\tau'$ . Then, he should strictly prefer partition  $\tau$  over  $\tau'$ .

**Proposition 8** *Let  $\preceq$  be a preference relation over  $\mathcal{P}(\Omega)$ . If  $P$  is broad,  $U(B, a)$  is continuous for all  $B$ , and  $\preceq$  satisfies dominance, then  $\preceq$  is not representable.*

**Remark.** If dominance is strengthened so that the conclusion follows without requiring  $p(\omega) > 0$ , then we obtain non-representation also for expected utility. We use maxmin as a natural way of incorporating multiple priors, and thus an uncountable number of atoms.

## 6 Concluding Remarks

In large sets, the representation of a decision maker's (DM) preferences by a utility depends on the "size" of her indifference curves. At one extreme, if DM is indifferent between all possible states her preferences are trivially representable; this is also the case if DM has a finite number of indifference curves. Preferences over information are typically weakly monotone, in the sense that more information is weakly preferred to less. We show that if indifference is ruled out for a large enough set of states by requiring strict monotonicity, there is no utility representation for preferences over information.

The question of weak vs. strict monotonicity is reminiscent of preferences over sequences of outcomes in repeated games. The "overtaking criterion" assumes that no outcome in an individual time period is important, while the "discounting criterion" assumes that a change in payoffs in any single time period makes a difference. Here, as in repeated games, both assumptions have their merit. But, unlike in repeated games, here they give very different conclusions. When individual states are unimportant (e.g. because of Savage's axioms, or because there are few alternative actions) there is a utility, but when enough states are important there is none. In our opinion this implies that any representation of preferences over information is not robust to changes in the environment.

## 7 Appendix

**Proof of Theorem 1.** Let  $\leq$  linearly order  $\Omega$  (such an order exists, for example, let  $\leq$  well order  $\Omega$ ). For all  $\omega \in \Omega$ , define  $\tau_\omega, \tau'_\omega \in \mathcal{P}(\Omega)$  by

$$\begin{aligned}\tau_\omega &= \{\{\theta\} : 0 \leq \theta < \omega\} \cup \{\theta : \omega \leq \theta\}, \\ \tau'_\omega &= \{\{\theta\} : 0 \leq \theta \leq \omega\} \cup \{\theta : \omega < \theta\}.\end{aligned}$$

Note that  $\tau'_\omega$  is finer than  $\tau_\omega$ , and that if  $\omega < \hat{\omega}$ , then  $\tau'_\omega \preceq \tau_{\hat{\omega}}$ .

Suppose, by way of contradiction, that there is a utility  $u : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$  that represents  $\preceq$ . Then, for each  $\omega \in \Omega$  there is a rational number  $r(\omega)$  such that  $u(\tau_\omega) < r(\omega) < u(\tau'_\omega)$ . Let  $\omega \neq \hat{\omega}$ , say  $\omega < \hat{\omega}$ , then  $r(\omega) < u(\tau'_\omega) \leq u(\tau_{\hat{\omega}}) < r(\hat{\omega})$ . Thus  $r : \Omega \rightarrow \mathbf{Q}$  is an injection, a contradiction as  $\Omega$  is uncountable. ■

**Proof of Theorem 2.** Let  $\leq$  linearly order  $\Omega$ , and endow  $\Omega$  with the order-interval topology. For all  $\omega \in \Omega$ , let  $\mathcal{B}_\omega$  denote the Borel sigma algebra on  $\{\theta : \theta < \omega\}$ , and  $\mathcal{B}'_\omega$  the Borel  $\sigma$  algebra on  $\{\theta : \theta \leq \omega\}$ . To each  $\omega$  we associate two  $\sigma$  algebras  $\sigma_\omega$  and  $\sigma'_\omega$  defined by

$$\begin{aligned}\sigma_\omega &= \mathcal{B}_\omega \cup \{B \cup \{\theta : \omega \leq \theta\} : B \in \mathcal{B}_\omega\} \\ \sigma'_\omega &= \mathcal{B}'_\omega \cup \{B \cup \{\theta : \omega < \theta\} : B \in \mathcal{B}'_\omega\}\end{aligned}$$

First, it is easy to check that  $\sigma_\omega$  and  $\sigma'_\omega$  are indeed  $\sigma$  algebras. Second,  $\sigma_\omega \subseteq \sigma'_\omega$ , as any  $\{\theta : \theta < \omega\}$ -open set is open and contained in  $\{\theta : \theta \leq \omega\}$ . Then,  $\{\omega\} \in \sigma'_\omega$  and  $\{\omega\} \notin \sigma_\omega$  imply that  $\sigma_\omega \subsetneq \sigma'_\omega$ .

Suppose, by way of contradiction, that there is a utility  $u : \mathcal{F}(\Omega) \rightarrow \mathbf{R}$  that represents  $\preceq$ . Monotonicity ensures that one can assign to each  $\omega$  a rational  $r(\omega)$  such that

$$u(\sigma_\omega) < r(\omega) < u(\sigma'_\omega).$$

Now pick any  $\beta \in \Omega$ , say  $\omega < \beta$ . Since any  $\{\theta : \theta \leq \omega\}$ -closed set is  $\{\theta : \theta < \beta\}$ -closed,  $\mathcal{B}'_\omega \subseteq \mathcal{B}_\beta$ . Then,  $\{\theta : \omega < \theta \leq \beta\} \in \mathcal{B}_\beta$  implies that  $\sigma'_\omega \subseteq \sigma_\beta$ . Thus,

$$u(\sigma_\omega) < r(\omega) < u(\sigma'_\omega) \leq u(\sigma_\beta) < r(\beta) < u(\sigma'_\beta),$$

and  $r$  is injective, a contradiction. ■

**Proof of Theorem 5.** The proof makes use of the classical representation theorem of Garrett Birkhoff (see Theorem 3.5 in Kreps (1988)): a preference relation  $\preceq$  on a choice space  $X$  is representable if and only if  $X$  is order-separable; that is, if and only if there is  $Z \subseteq X$ , countable, such that  $x, y \in X$ ,  $x \prec y$  imply that there is  $z \in Z$  with  $x \preceq z \preceq y$ .

(if) If  $\Omega$  is finite, then  $\mathcal{P}(\Omega)$  is finite and therefore order-separable. By Birkoff's Theorem,  $\preceq$  is representable.

(only if) Let  $\Omega$  be infinite and  $\omega \in \Omega$  an atom for  $\preceq$ . There is an uncountable number of sets  $A$  that contain  $\omega$  and have at least another element. Let  $p(A) = \{A, A^c\}$  and

$p'(A) = \{\{\omega\}, A \setminus \{\omega\}, A^c\}$ . Note that  $p(A), p'(A) \in \mathcal{P}(\Omega)$  and that  $p(A) \prec p'(A)$ . Also note that there is no  $x \in \mathcal{P}(\Omega)$  with  $p(A) \prec x \prec p'(A)$ . Order-separability would require that there be  $z \in Z$  with  $p(A) \preceq z \preceq p'(A)$  i.e. that either  $p(A)$  or  $p'(A)$  be in  $Z$ . Since  $\Omega$  is not finite, it has uncountably many subsets like  $A$ , hence  $Z$  could not be countable. By Birkhoff's theorem, there is no utility representation. ■

**Proof of Theorem 7.** For any ordered set  $\langle X, \preceq \rangle$ , let  $(x, y) \equiv \{z \in X : x \prec z \prec y\}$ , for  $x, y$  in  $X$ .

Let  $\leq$  well-order  $\Omega$ . We shall construct an uncountable collection of intervals in  $\mathcal{P}(\Omega)$ . Let

$$\begin{aligned}\tau_\omega &= \{\{\theta\} : \theta < \omega\} \cup \{\theta : \omega \leq \theta\}, \\ \tau &= \{\{\omega\} : \omega \in \Omega\}.\end{aligned}$$

Since  $\preceq$  is monotonic, for all  $\omega < \theta$ ,  $\tau_\omega \prec \tau_\theta$ . The collection of intervals  $\{(\tau_\omega, \tau)\}_{\omega \in \Omega}$  is well ordered by set inclusion, as  $\Omega$  is well ordered. Theorem 3.1 of Beardon et al. (2000) then ensures that  $\langle \mathcal{P}(\Omega), \preceq \rangle$  is long. ■

**Proof of Proposition 6.** Let  $C(\Omega, A)$  denote the space of continuous functions from  $\Omega$  to  $A$ , endowed with the topology of uniform convergence. If  $A$  is finite and  $\Omega$  a compact metric space,  $C(\Omega, A)$  is a separable metric space (see Aliprantis and Border (1999, Theorem 3.85)). Since separability is hereditary in metric spaces,  $f(\mathcal{P}(\Omega))$  is a separable metric space. Thus, by Debreu (1954, Theorem II) any continuous preference relation on  $f(\mathcal{P}(\Omega))$  is representable.

Let the preference relation  $\preceq$  on  $f(\mathcal{P}(\Omega))$  be defined by  $a_\tau \preceq a_{\tau'}$  if and only if  $\tau \preceq \tau'$ . A convergent sequence in  $C(\Omega, A)$  is eventually constant, as  $A$  is finite and  $C(\Omega, A)$  is endowed with the topology of uniform convergence. Thus  $\preceq$  is continuous. By Debreu (1954, Theorem II) there is a utility function  $u : f(\mathcal{P}(\Omega)) \rightarrow \mathbf{R}$  that represents  $\preceq$ . Defining  $v : \mathcal{P}(\Omega) \rightarrow \mathbf{R}$  by  $v(\tau) = u(f(\tau))$  we see that  $v$  represents  $\preceq$ . ■

**Proof of Proposition 8.** We now show that, if a partition  $\tau$  is a “one-point refinement” of  $\tau'$ , then  $\tau \succ \tau'$ . Pick any  $k \in \tau'$  with at least two elements, and fix  $\omega \in k$  with  $\omega \neq a[k]$ . We will now show that

$$\tau = \{l \cap \{\omega\} : l \in \tau'\} \cup \{l \cap \{\omega\}^c : l \in \tau'\}$$

Pareto Dominates  $\tau'$  ( $\tau$  is a one point refinement of  $\tau'$ ).

Note that for all  $\omega' \notin k_{\tau'}(\omega)$ , we have that  $k_{\tau'}(\omega') = k_\tau(\omega')$  and thus

$$U(k_\tau(\omega') \cap k_{\tau'}(\omega'), a[k_\tau(\omega')]) = U(k_\tau(\omega') \cap k_{\tau'}(\omega'), a[k_{\tau'}(\omega')])$$

Now, fix any  $\omega' \in k$ . Two cases must be considered.

I)  $\omega' = \omega$ . In this case,  $k_\tau(\omega') \cap k_{\tau'}(\omega') = \{\omega\}$ . Since  $P$  is broad there is  $p \in P$  with  $p(\{\omega\}) > 0$ . Since  $\omega \neq a[k]$  we have

$$\begin{aligned} U(\{\omega\}, a[k_\tau(\omega)]) &= U(\{\omega\}, a[k]) \\ &\leq \int_{\{\omega\}} \frac{u(\tilde{\omega}, a[k])}{p(\{\omega\})} dp(\tilde{\omega}) \\ &= u(\{\omega\}, a[k]) \\ &< u(\omega, \omega) = U(\{\omega\}, a[k_\tau(\omega)]) \end{aligned}$$

II)  $\omega' \neq \omega$ . In this case,  $k_\tau(\omega') \cap k_{\tau'}(\omega') = k_\tau(\omega')$ . Then, by definition  $U(k_\tau(\omega'), a[k_\tau(\omega')]) \geq U(k_\tau(\omega'), a[k_{\tau'}(\omega')])$ .

In fact, monotonicity to one-point refinements is all that is needed in the proof of Theorem 1. Thus  $\preceq$  is not representable. ■

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